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# Spiky strings in $\kappa$ -deformed AdS

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**ABSTRACT:** We study rigidly rotating strings in  $\kappa$ -deformed AdS background. We probe this classically integrable background with ‘spiky’ strings and analyze the string profiles in the large charge limit systematically. We also discuss the dispersion relation among the conserved charges for these solutions in long string limit.

**KEYWORDS:** AdS-CFT Correspondence, Bosonic Strings

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## 1 Introduction

Integrability has been proved to be one of the most useful tools in studying string spectrum in various semisymmetric superspaces [1].<sup>1</sup> The integrable structure on both sides of the celebrated AdS/CFT correspondence [3] has opened up new areas in the study of string theory, among others, for example the type IIB superstring theory on  $\text{AdS}_5 \times S^5$  has been shown to be described as supercoset sigma model [4] and integrability of AdS has been explored in [5]. It has been shown that in the large angular momentum or large R-charge limit, both sides of the AdS/CFT duality are easier to probe [6–10]. It was further noticed that the spectrum of semiclassical spinning strings in this limit can be related to the anomalous dimension of the  $\mathcal{N} = 4$  Super Yang-Mills (SYM), and this can be derived from the relation between conserved charges of the rigidly rotating strings in  $\text{AdS}_5 \times S^5$ . This comes from the amazing proposal of [11, 12] which relates the dilatation operator in  $\mathcal{N} = 4$  Super Yang-Mills to the Hamiltonian of an integrable Heisenberg  $\text{SU}(2)$  type spin-chain system. Inspired by this, many studies of rotating strings on AdS and asymptotically AdS spaces have come up, for example in [13–19].

The integrable system description of this duality opens up possibility of illuminating the gauge theories dual to integrable deformations of the  $\text{AdS}_5 \times S^5$  case. For many cases, these kind of deformations have been achieved by T-dualizing the existing low-dimensional bosonic sigma models [20–25]. Classical strings on these backgrounds are also studied in, for example, [26, 27]. Contrary to these models, a novel one-parameter deformation of the

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<sup>1</sup>For an introduction to integrability in AdS/CFT one can see [2].

bosonic sigma model was constructed in [28] following a few earlier proposals on Yang-Baxter sigma models put forward by Klimcik [29–31].<sup>2</sup> This real deformation parameter is often called  $\varkappa$  or  $\eta$  with  $\varkappa = \frac{2\eta}{1-\eta^2}$ , where  $\varkappa \in [0, \infty)$ . The foundations of the integrable structure of string theory as described by this deformed sigma model have been presented in [34–40].

The new deformed model has the bosonic symmetry group reduced from  $SO(2, 4) \times SO(6)$  for  $AdS_5 \times S^5$  to its Cartan subgroup  $[U(1)]^3 \times [U(1)]^3$ , which makes most of the symmetry of the original space hidden or sometimes called “ $q$ -deformed”. Indeed the deformed integrable sigma model has a  $q$ -deformed  $PSU(2, 2|4)$  as the symmetry group [38], with  $q$  as a function of  $\eta$ . This model seemingly has no manifest space-time supersymmetry but the  $q$ -deformed symmetry group suggests that the space-time supersymmetry can be represented in a non-trivial way. The exact 10d metric and the associated Neveu-Schwarz B-field were found in [34] and various consistent truncations were also discussed in [41]. The corresponding deformed full type IIB supergravity solutions for the subset  $AdS_2 \times S^2$  and  $AdS_3 \times S^3$  have been found out recently in [42]. It has been shown that the solutions depend on a free parameter which in turn is a function of the deformation parameter  $\varkappa$ . In a related development, the two parameter and three parameter generalizations of this deformed supercoset sigma model have been outlined in [43]. It seems that there is a singularity surface associated with these deformed background metrics, which was addressed in detail in [44]. Various minimal surfaces and cusped Wilson loops in this background were studied in [45, 46]. Also more recently, some three point correlation functions have been studied in [47]. The gauge theory dual of this solution, if any, needs to be found fully.

Now, given the integrable nature of this deformation, it is natural to look for rigidly rotating string solutions and explore whether there can be a spin-chain like underlying system or not. In this connection, in a subspace of the deformed  $AdS_5 \times S^5$ , the so called giant magnon and single spike solutions of the string have been studied along with the finite size corrections [36, 44, 48–50]. Despite the complex structure of the giant magnon solution on a deformed  $\mathbb{R} \times S^2$  subspace, it clearly had retained the periodicity in magnon momenta  $p$ . It was also shown in the limit  $\varkappa \rightarrow 0$ , the relation reduces to the form of usual giant magnon dispersion relation proposed by Hofman-Maldacena (HM) [51]. Also it was shown in [50] that the usual single spike strings [52] and the giant magnon solutions on the deformed sphere can be derived as two limits of a single setup. The deformed Neumann and the Neumann-Rosochatius systems for spinning-strings have been described in [44, 53]. On the deformed AdS, the usual folded GKP like string solutions [54] were discussed in [44] and it was shown that in the ‘long’ string limit, the expression for cusp anomalous dimension does not reduce to the undeformed AdS case even if one takes the limit  $\varkappa \rightarrow 0$ . Although in the investigation of ‘pulsating’ strings on  $(AdS)_\varkappa$  [55] no such inconsistencies were found.

In the present paper, we focus on the ‘spiky’ string profiles having multiple cusps in the  $\varkappa$  deformed AdS by probing a classical rotating string. The symmetric spiky strings

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<sup>2</sup>Yang-Baxter sigma models based on classical Yang-Baxter equation (CYBE) were recently studied at length in, e.g. [32]. In [33] it was further shown that the TsT type deformations mentioned earlier also belong to the class of Yang-Baxter type deformations.

on  $\text{AdS}_3$  were first studied in [56] from the worldsheet viewpoint. The usual dispersion relations for ‘long’  $\text{AdS}$  spiky strings were constructed in [56, 57]. A spike (or cusp) is defined as a discontinuity in the spacelike unit tangent vector to the string, which can also be present on a smooth worldsheet. They correspond to single trace operators in the dual gauge theories. Also more general spiky string solutions in  $\text{AdS}$  were described in [58] in terms of a Polyakov string with a generalized embedding ansatz for the open string.<sup>3</sup>

The rest of the paper is organized as follows. In section 1, we describe rigidly rotating open string profiles with a general embedding ansatz. After revisiting the string solutions in [58] completely and systematically classifying the profiles in various regions of the parameter space, we discuss the same in the  $\varkappa$  deformed  $\text{AdS}_3 \times S^1$ . We discuss the effect of turning on the deformation parameter on the string profiles. We also write an approximate dispersion relation for the 2-spin spiky strings. In section 2, we discuss the particular kind of closed spiky strings studied in [56] in this deformed background and plot the symmetric string profiles for various values of the parameters involved. We also find out the behaviour of the conserved charges for each spike in the large spin limit. This reduces to the same expression for  $E - S$  described in [60] in the GKP limit, at least in the leading order approximation. Finally in section 3, we present our conclusions and outlook.

## 2 Polyakov strings and Spikes in $(\text{AdS})_\varkappa$

In this subsection we mainly discuss about rigidly rotating open strings in the lines of [58]. After revisiting the original string solutions in  $\text{AdS}_3 \times S^1$ , we will generalize it to one parameter deformed geometry.

### 2.1 Classifying open strings in $\text{AdS}_3 \times S^1$

In this section we review the main features of the multispin giant magnon-like solutions in undeformed  $\text{AdS}$  as discussed in [58] and present visual description of the solutions. For this we start with the usual  $\text{AdS}_3 \times S^3$  background given by the following metric

$$ds^2 = -\cosh^2 \chi dt^2 + d\chi^2 + \sinh^2 \chi d\phi^2 + \sin^2 \theta d\varphi^2 + d\theta^2 + \cos^2 \theta d\psi^2 \quad (2.1)$$

Since  $\theta = \frac{\pi}{2}$  is a viable solution for the classical string equations of motion for this background, we can write the usual  $\text{AdS}_3 \times S^1$  metric as

$$ds^2 = -\cosh^2 \chi dt^2 + d\chi^2 + \sinh^2 \chi d\phi^2 + d\varphi^2. \quad (2.2)$$

To describe the 2-spin giant magnon/spiky string solution we take the rotating string embedding ansatz as follows

$$t = \tau + g_1(y), \quad \chi = \chi(y), \quad \phi = \omega(\tau + g_2(y)), \quad \varphi = \mu\tau, \quad (2.3)$$

where  $y = \sigma - v\tau$  and  $0 < v < 1$ . The polyakov action of the F-string is given as

$$S = -\frac{T}{2} \int d\sigma d\tau [\sqrt{-\gamma} \gamma^{\alpha\beta} g_{MN} \partial_\alpha X^M \partial_\beta X^N]. \quad (2.4)$$

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<sup>3</sup>It is worth mentioning here that for a nice exposure to integrability in  $\text{AdS}_3/\text{CFT}_2$  correspondence one might go through [59] and references therein.

Variation of the action (2.4) with respect to  $X^M$  gives us the following equation of motion

$$2\partial_\alpha(\eta^{\alpha\beta}\partial_\beta X^N g_{KN}) - \eta^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N \partial_K g_{MN} = 0, \quad (2.5)$$

and variation with respect to the metric gives the two Virasoro constraints,

$$\begin{aligned} g_{MN}(\partial_\tau X^M \partial_\tau X^N + \partial_\sigma X^M \partial_\sigma X^N) &= 0, \\ g_{MN}(\partial_\tau X^M \partial_\sigma X^N) &= 0. \end{aligned} \quad (2.6)$$

Now the equations of motion for  $t$  and  $\phi$  gives

$$\partial_y g_1 = \frac{1}{1-v^2} \left[ \frac{C_1}{\cosh^2 \chi} - v \right], \quad \partial_y g_2 = \frac{1}{1-v^2} \left[ \frac{C_1}{\sinh^2 \chi} - v \right]. \quad (2.7)$$

Also from the  $\chi$  equation, supplemented by the boundary condition  $\chi'(y) \rightarrow 0$  as  $\chi \rightarrow 0$ , we get the condition  $C_2 = 0$ . The equation of motion then can be written as

$$\chi'(y) = \pm \frac{\sqrt{1-\omega^2}}{(1-v^2)} \tanh \chi \sqrt{\cosh^2 \chi - \frac{C_1^2}{1-\omega^2}}. \quad (2.8)$$

Again subtracting the two Virasoro constraints we get a relation between the various constants of the embedding ansatz

$$\mu^2 - \frac{C_1}{v} = 0. \quad (2.9)$$

Also by equating the  $\chi$  equation with the first Virasoro constraint we get the equation for  $C_1$  as

$$C_1^2 - C_1 \frac{1+v^2}{v} + 1 = 0, \quad (2.10)$$

the solutions of which gives the values of  $C_1$  consistent with the Virasoro constraints. We can see the roots of the above equation correspond to two different solutions of the string equation of motion

$$\begin{aligned} C_1 &= v \\ &= \frac{1}{v}. \end{aligned} \quad (2.11)$$

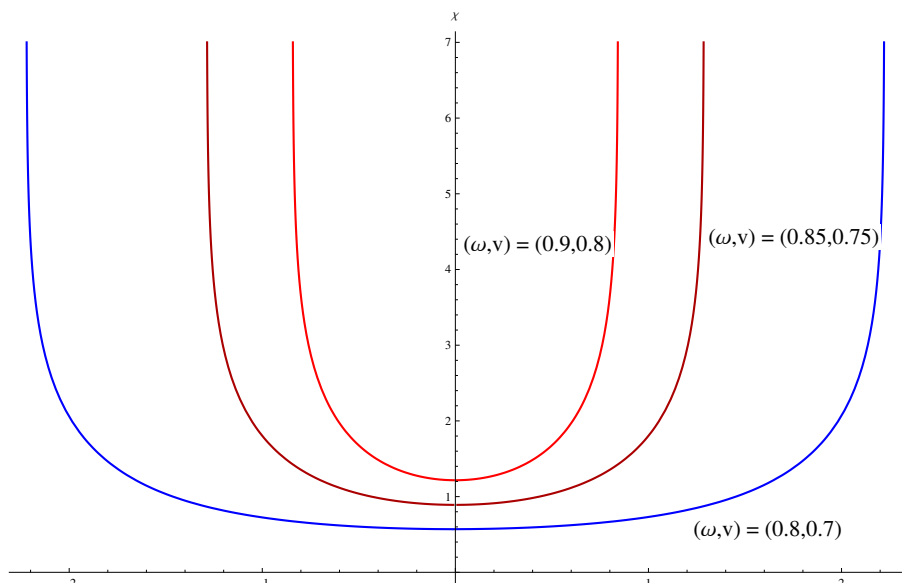
Since for the case  $C_1 = \frac{1}{v}$  we would have  $\mu = \frac{1}{v}$  then it is a natural choice to work with  $C_1 = v$  as detailed in [58]. However, we will discuss qualitative features of both the cases. Also note that the choice of the constant is supported by demanding forward propagation of strings in this case, i.e.

$$t = \frac{1}{1-v^2} \left( 1 - \frac{v^2}{\cosh^2 \chi} \right) > 0. \quad (2.12)$$

Now, integrating (2.8) we can write down the string profile as

$$y = \pm \frac{(1-v^2)}{\sqrt{1-\omega^2}} \frac{1}{\sqrt{1-\alpha^2}} \tanh^{-1} \left[ \sqrt{\frac{\cosh^2 \chi - \alpha^2}{1-\alpha^2}} \right], \quad (2.13)$$

where  $\alpha^2 = \frac{v^2}{1-\omega^2} = \cosh^2 \chi_1$  (say) is a root of the equation  $\chi'(y) = 0$ . As discussed in [58, 61] depending on the value of this root, we can have two different classes of string solutions for the system.



**Figure 1.** Hanging string profiles for various values of  $(\omega, v)$ .

- i) When  $\cosh^2 \chi_1 > 1$  the solution can be rewritten in the form

$$y = \pm \frac{(1 - v^2)}{\sqrt{v^2 + \omega^2 - 1}} \cos^{-1} \left( \frac{\sqrt{v^2 + \omega^2 - 1}}{\sqrt{1 - \omega^2} \sinh \chi} \right). \quad (2.14)$$

It is clear from here that  $\frac{-\pi}{2\beta} \leq y \leq \frac{\pi}{2\beta}$ , where  $\beta = \frac{\sqrt{v^2 + \omega^2 - 1}}{(1 - v^2)}$ . These string profiles correspond to the so called hanging strings as shown in figure 1. It can be seen that as  $\beta(\omega, v)$  increases, the width of the string decreases as range of  $y$  becomes smaller.

- ii) When  $\cosh^2 \chi_1 \leq 1$  the solution can be rearranged in the expression

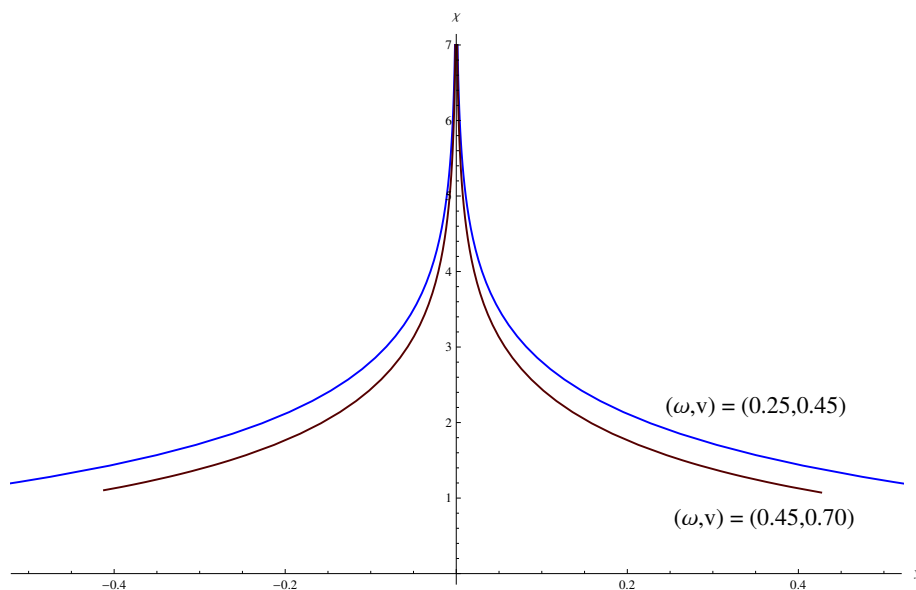
$$y = \pm \frac{(1 - v^2)}{\sqrt{1 - v^2 - \omega^2}} \sinh^{-1} \left( \frac{\sqrt{1 - v^2 - \omega^2}}{\sqrt{1 - \omega^2} \sinh \chi} \right) \quad (2.15)$$

Here the range of  $y$  changes to  $-\infty \leq y \leq \infty$  and these are the spiky string profiles as shown in figure 2. The solution is supported in the infinite range of  $y$  which makes it more interesting to study. One can see from the figures how the string profiles vary under various values of  $(\omega, v)$ .

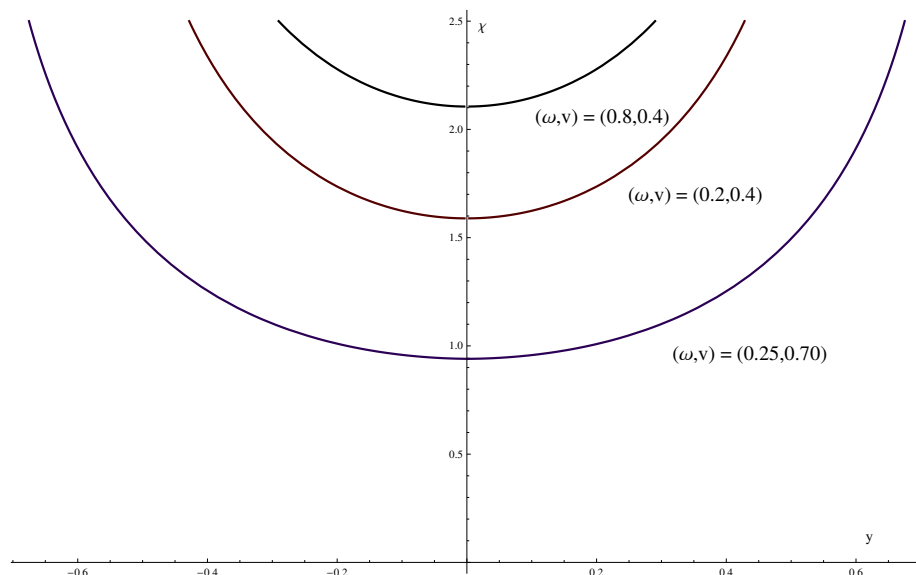
However, there is another choice of  $C_1 = \frac{1}{v}$ , which leads to the following string equation of motion

$$\chi'(y) = \pm \frac{\sqrt{1 - \omega^2}}{(1 - v^2)} \tanh \chi \sqrt{\cosh^2 \chi - \frac{1}{v^2(1 - \omega^2)}} \quad (2.16)$$

Since  $0 < \omega^2 < 1$  and  $0 < v^2 < 1$ , we can readily show that  $\frac{1}{v^2(1 - \omega^2)} > 1$  is always true. Then it is evident that we will only get one kind of string solutions, namely the hanging ones. The string profiles are plotted in figure 3 for various values of  $(\omega, v)$ .



**Figure 2.** Spiky string profiles for various values of  $(\omega, v)$ .



**Figure 3.** Hanging strings for the choice of parameter  $C_1 = \frac{1}{v}$  for various  $(\omega, v)$ .

## 2.2 Open strings in $(\text{AdS}_3 \times S^1)_\kappa$

To start with we write the total deformed  $\text{AdS}_3 \times S^3$  metric which has been discussed as a consistent truncation of one parameter deformed family of  $\text{AdS}_5 \times S^5$  in [34, 41].

$$\begin{aligned}
 ds^2 = & \frac{1}{1 - \kappa^2 \sinh^2 \rho} \left[ -\cosh^2 \rho dt^2 + d\rho^2 \right] + \sinh^2 \rho d\phi^2 \\
 & + \frac{1}{1 + \kappa^2 \cos^2 \theta} \left[ \sin^2 \theta d\varphi^2 + d\theta^2 \right] + \cos^2 \theta d\psi^2
 \end{aligned} \tag{2.17}$$

Now we can see that there is a singularity in the metric for the value  $\rho = \rho_s = \sinh^{-1} \frac{1}{\kappa}$ , which also manifests itself in the scalar curvature. However we can get rid of this problem by mapping into a good coordinate system which describes only the inside of this so-called ‘singularity surface’. This transformation has been explicitly discussed in [44] in the following form,

$$\frac{\cosh \rho}{\sqrt{1 - \kappa^2 \sinh^2 \rho}} = \cosh \chi, \quad (2.18)$$

where the range of the AdS radius is mapped from  $\rho \in [0, \sinh^{-1} \frac{1}{\kappa})$  to  $\chi \in [0, \infty)$  and ‘zooms’ into the region inside the singularity surface. In [44] it was conjectured that this singularity surface might act as a holographic screen inside of which all the degrees of freedom are confined. In what follows, we will mainly discuss spiky string solutions which approach  $\rho = \rho_s$  instead of  $\rho \rightarrow \infty$ . Now as it can be shown that  $\theta = \frac{\pi}{2}$  is still a solution of the classical string equation of motion, we can write the transformed  $\text{AdS}_3 \times S^1$  part of the deformed metric as,

$$ds^2 = \frac{1}{1 + \kappa^2 \cosh^2 \chi} [\sinh^2 \chi d\phi^2 + d\chi^2] - \cosh^2 \chi dt^2 + d\varphi^2. \quad (2.19)$$

To study rigidly rotating open strings in the above background, we again use similar ansatz as in the previous section

$$t = \tau + h_1(y), \quad \chi = \chi(y), \quad \phi = \omega(\tau + h_2(y)), \quad \varphi = \Omega \tau. \quad (2.20)$$

Then from the  $t$  and  $\phi$  equations of motion we would get,

$$\partial_y h_1 = \frac{1}{1 - v^2} \left[ \frac{A_1}{\cosh^2 \chi} - v \right], \quad \partial_y h_2 = \frac{1}{1 - v^2} \left[ \frac{A_2(1 + \kappa^2 \cosh^2 \chi)}{\sinh^2 \chi} - v \right]. \quad (2.21)$$

As before, the  $\chi$  equation supplemented by the boundary condition  $\chi'(y) \rightarrow 0$  as  $\chi \rightarrow 0$  we conclude that  $A_2 = 0$ . Thus the form of the  $\chi$  equation becomes

$$\chi'(y) = \pm \frac{\tanh \chi}{1 - v^2} \sqrt{f(\chi)}, \quad (2.22)$$

where  $f(\chi) = \kappa^2 \cosh^4 \chi + (1 - \omega^2 - A_1^2 \kappa^2) \cosh^2 \chi - A_1^2$ . Again the Virasoro constraints give the same results as the previous section when subtracted. We will choose  $A_1 = v$  for further calculations.

### 2.2.1 String profiles and the effect of $\kappa$

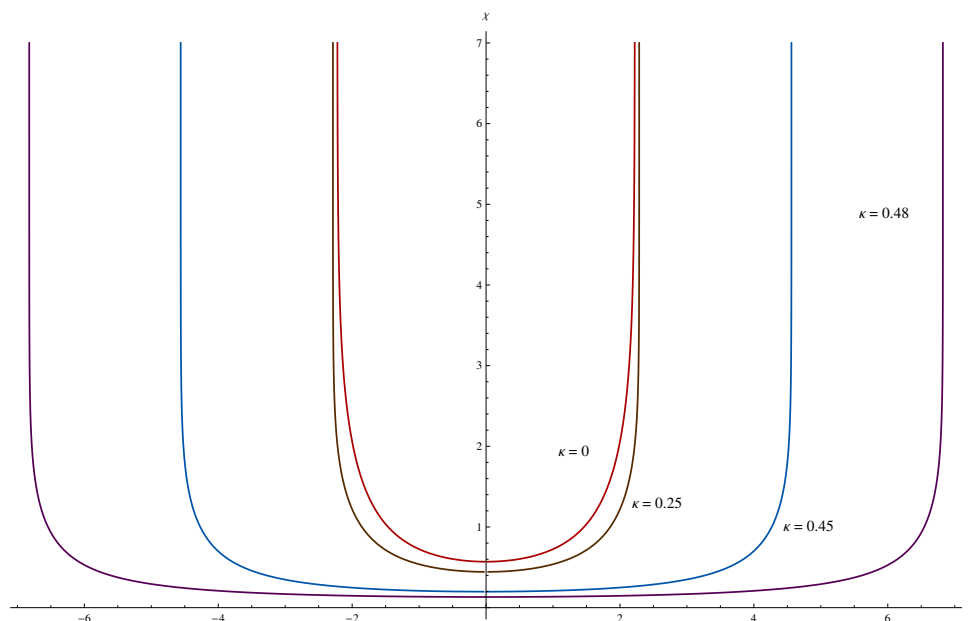
To find the rotating string profile we have to integrate equation (2.22) between some appropriate range. We note that since  $f(\chi)$  is quadratic function in  $\cosh^2 \chi$ , for the parameter space  $(\omega, v) \in (0, 1)$  and  $\kappa \in [0, \infty)$  we can always find two real roots of the function,

$$\cosh^2 \chi_{\pm} = \frac{-1 + \omega^2 + v^2 \kappa^2 \pm \sqrt{(-1 + \omega^2 + v^2 \kappa^2)^2 + 4v^2 \kappa^2}}{2\kappa^2}, \quad (2.23)$$

where  $\chi_+ > \chi_-$ . The string profile can then be written in the integral form

$$y = \pm \int \frac{(1 - v^2)}{\kappa \tanh \chi \sqrt{(\cosh^2 \chi - \cosh^2 \chi_+)(\cosh^2 \chi - \cosh^2 \chi_-)}} d\chi. \quad (2.24)$$

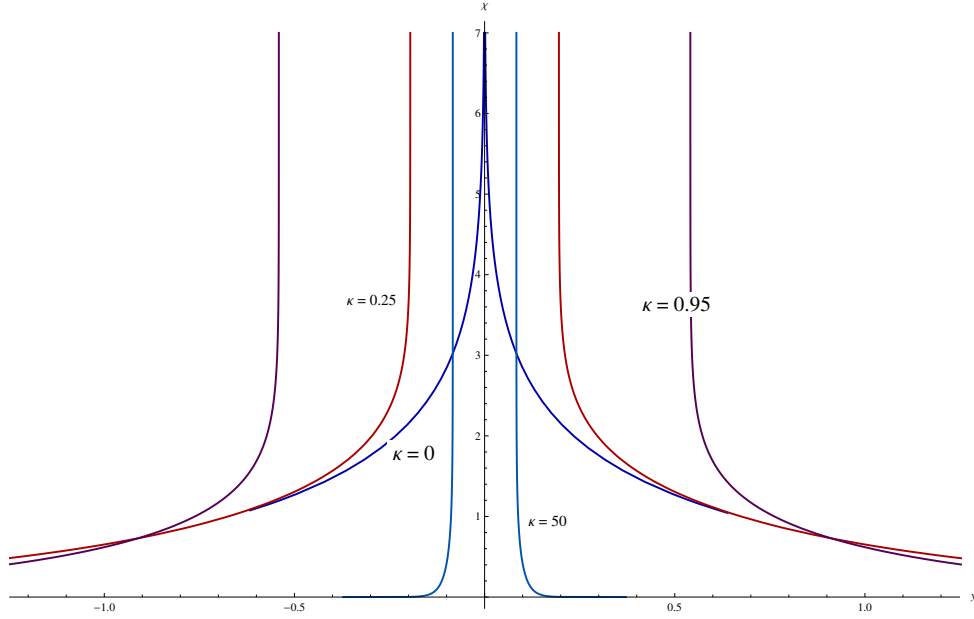




**Figure 4.** Hanging string profiles for various values of  $\varkappa$  with fixed  $(\omega, v) = (0.8, 0.7)$ . Notice the flattening of the profile as  $\varkappa$  is increased.

Now the string solutions can exist only in the cases  $f(\chi) \geq 0$ . This can only be possible if either  $\chi \geq \chi_+$  or  $\chi \leq \chi_-$ . It can be clearly seen that the  $\varkappa = 0$  string profiles are not smoothly connected to the finite  $\varkappa$  profiles. However it is notable that for a fixed  $\varkappa$  the above expression depends on the parameters  $(\omega, v)$  only. The solutions can be classified by comparison of  $\chi_+$  and  $\chi_-$ . Again a simple analysis shows that there are two kinds of string solution possible, the hanging strings and the spiky strings. Spiky strings extend all the way upto the asymptotic infinity and are the solutions we are interested in.

- i) Firstly, it can be clearly seen that  $\cosh^2 \chi_-$  is always negative for  $(\omega, v, \varkappa) \in \mathcal{R}$ . The hanging string profiles exist for the region where  $\cosh^2 \chi_+ \geq 1$ . For example using  $(\omega, v) = (0.8, 0.7)$  this inequality is violated at  $\varkappa \simeq 0.503$ . As  $\varkappa$  increases the hanging strings are more and more flattened as the range of  $y$  increases. This case is depicted in figure 4.
- ii) The second case is when  $\cosh^2 \chi_+ < 1$ , which give rise to the spiky strings. Note that in this region the value of the root remains almost constant for the changing values of  $\varkappa$ . It can be seen from figure 5 that as  $\varkappa$  increases the spiky ‘cusp’ nature of the strings are destroyed. For large values of the deformation parameter, the strings simply become parallel to the  $\chi$  axis.



**Figure 5.** Spiky string profiles for various values of  $\varkappa$  for fixed  $(\omega, v) = (0.2, 0.4)$ .

### 2.2.2 Conserved charges and dispersion relation

We can construct the following conserved charges for the rotating string solutions looking at the symmetries of the background. They are

$$\begin{aligned} E &= - \int \frac{\partial \mathcal{L}}{\partial t} d\sigma = \frac{2\hat{T}}{(1-v^2)} \int (\cosh^2 \chi - v^2) d\sigma, \\ S &= \int \frac{\partial \mathcal{L}}{\partial \dot{\phi}} d\sigma = 2\hat{T} \frac{\omega}{(1-v^2)} \int \frac{\sinh^2 \chi}{1 + \varkappa^2 \cosh^2 \chi} d\sigma, \\ J &= \int \frac{\partial \mathcal{L}}{\partial \dot{\phi}} d\sigma = 2\hat{T}\Omega \int d\sigma, \end{aligned} \quad (2.25)$$

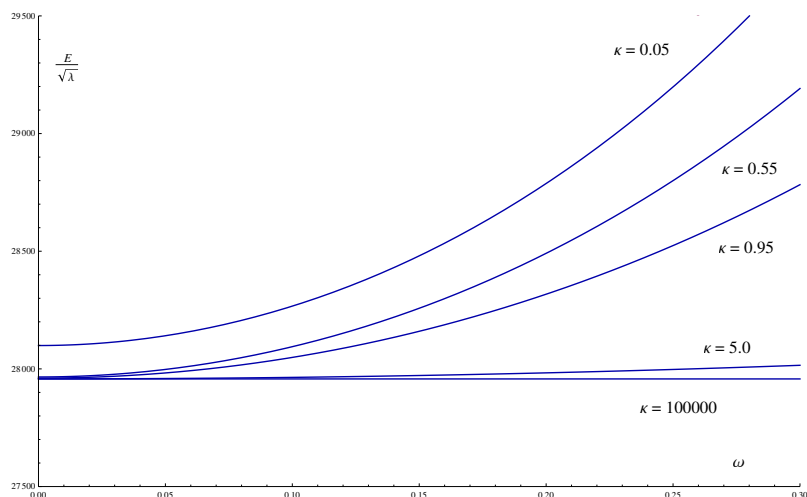
where  $\hat{T} = \frac{\sqrt{\lambda}}{2\pi} \sqrt{1 + \varkappa^2}$ . It can be seen that the above charges satisfy a relation

$$E - \frac{J}{\Omega} = \frac{S}{\omega} + \mathcal{K}, \quad (2.26)$$

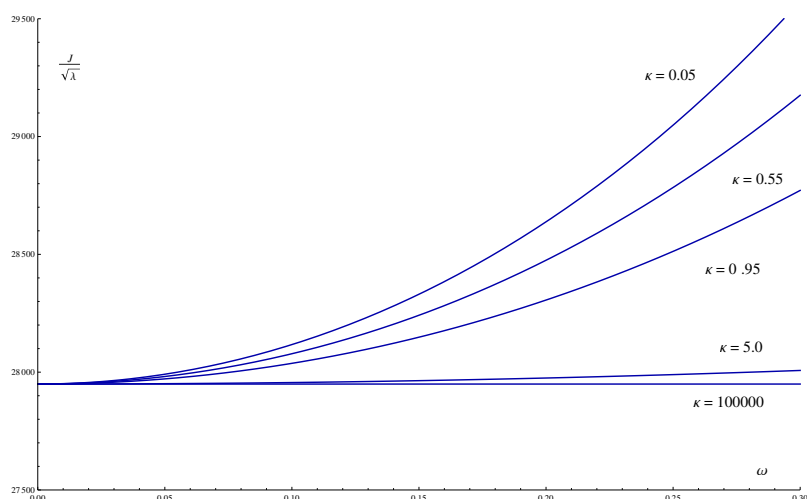
where  $\mathcal{K}$  acts like a correction term to the dispersion relation. Note that, all the charges themselves are dependent on  $\varkappa$ . The expression for  $\mathcal{K}$  given by

$$\mathcal{K} = \frac{2\hat{T}}{(1-v^2)} \int \frac{\varkappa^2 \sinh^2 \chi \cosh^2 \chi}{1 + \varkappa^2 \cosh^2 \chi} d\sigma. \quad (2.27)$$

It can be explicitly checked that when we take the  $\varkappa = 0$  the correction piece goes to zero also and the dispersion relation reduces to the exact one  $E - \frac{J}{\Omega} = \frac{S}{\omega}$  as given in [58]. Now as the charges are integrated upto  $\chi = \infty$ , they diverge. Since the integrals involved are really complex, we can't obtain regularized expressions for  $E$ ,  $S$  and  $J$  analytically. Instead we focus on the nature of the charges as they change with the value of winding number



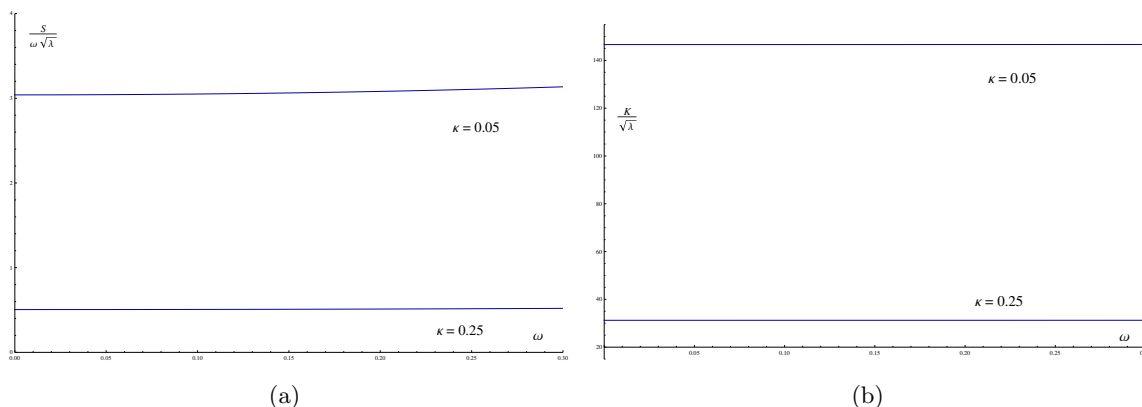
**Figure 6.** Energy of the rotating strings plotted against  $\omega$  for different values of  $\varkappa$ . For very large values of deformation we can see the Energy becomes more or less indifferent to the change in  $\omega$ . The cutoff is implemented at  $\chi = 50$ .



**Figure 7.** Angular momenta of the rotating strings coming from the sphere part plotted against  $\omega$  for different values of  $\varkappa$ . For very large values of deformation we can see it exactly behaves like the Energy. The cutoff is implemented at  $\chi = 50$ .

$\omega$ . For a numerical computation, we impose a cutoff at  $\chi = 50$  and plot the expressions in the figures 6 and 7. From the plot it can be seen that while  $E$  and  $J$  have large values for small winding number  $\omega$ , their difference remains small and finite.

Similarly we also show the behaviour of  $\frac{S}{\omega}$  and  $\mathcal{K}$  for various values of the winding number in figures 8(a) and 8(b). It is worth noting that both the expressions are extremely sensitive to  $\varkappa$  for fixed values of  $v$ . For a fixed  $\varkappa$  these expressions remain almost constant with changing  $\omega$ .



**Figure 8.** (a) Spin  $\frac{S}{\omega}$  of the rotating strings plotted against  $\omega$  for different values of  $\varkappa$  with  $v = 0.4$ . The cutoff is again at  $\chi_\Lambda = 50$ . (b)  $\mathcal{K}$  of the spiky strings plotted against  $\omega$  for different values of  $\varkappa$  with  $v = 0.4$ . The integration cutoff is at  $\chi_\Lambda = 50$ .

### 3 Closed strings and spikes in one parameter deformed AdS

It is worth mentioning, that closed strings in the  $\varkappa$  deformed AdS have been studied in [44, 60] in the GKP limit. The long string limit of the GKP solution (extending to the apparent boundary) in this case has been shown to be quite complex. One interesting feature is that the long string dispersion relation does not reduce to the usual scaling  $E - S \sim \log S$  for undeformed AdS, even when  $\varkappa \rightarrow 0$ . Also it has been shown that the classical spinning string can not be stretched beyond the singularity surface all the way to the real boundary. This prompts us to look for symmetric closed spiky string solutions first discussed in [56] for AdS<sub>3</sub> and see if they have the same kind of dynamics as the GKP ones. We will review the Nambu-Goto spiky strings in the simple AdS background and try to compare them with deformed AdS calculations.

#### 3.1 Review of Nambu-Goto spiky strings in AdS

Let us consider Nambu-Goto strings moving in the AdS<sub>3</sub> space parameterized by the metric

$$ds^2 = -\cosh^2 \chi dt^2 + d\chi^2 + \sinh^2 \chi d\phi^2, \quad (3.1)$$

with the following embedding ansatz,

$$t = \tau, \quad \phi = \omega\tau + \sigma, \quad \chi = \chi(\sigma). \quad (3.2)$$

The Nambu-Goto action for this fundamental string as

$$S = -\frac{\sqrt{\lambda}}{2\pi} \int \sqrt{-\dot{X}^2 X'^2 + (\dot{X} X')^2}. \quad (3.3)$$

The dots represent derivative w.r.t.  $\tau$  and primes represent derivatives w.r.t.  $\sigma$  for the notation of this section. Now following the equations of motion for  $t$  and  $\phi$  we can write

$$\frac{\sinh^2 \chi \cosh^2 \chi}{\sqrt{\chi'^2 (\cosh^2 \chi - \omega^2 \sinh^2 \chi) + \sinh^2 \chi \cosh^2 \chi}} = C$$

$$\Rightarrow \chi' = \frac{1}{2} \frac{\sinh 2\chi}{\sinh 2\chi_0} \frac{\sqrt{\sinh^2 2\chi - \sinh^2 2\chi_0}}{\sqrt{\cosh^2 \chi - \omega^2 \sinh^2 \chi}} \quad (3.4)$$

Here the integration constant is taken as  $C^2 = \frac{1}{4} \sinh^2 2\chi_0$ . Now the r.h.s. of the above equation blows up at  $\chi_1 = \coth^{-1} \omega$ , which gives rise to the spiky string solutions with the spike height  $\chi_1$ . The ‘valleys’ of the string profile occurs at the zeroes of  $\chi'$ , i.e. at  $\chi = \chi_0$ . Integrating the above equation, we can get the string profile for our rigidly rotating ansatz as the following,

$$\sigma = \frac{\sinh 2\chi_0}{\sqrt{2} \sqrt{\alpha_0 + \alpha_1} \sinh \chi_1} \left\{ \Pi \left( \frac{\alpha_1 - \alpha_0}{\alpha_1 - 1}, \Theta, \mathcal{Q} \right) - \Pi \left( \frac{\alpha_1 - \alpha_0}{\alpha_1 + 1}, \Theta, \mathcal{Q} \right) \right\}, \quad (3.5)$$

where  $\Pi$  is the standard elliptic integral of the third kind and

$$\mathcal{Q} = \frac{\alpha_1 - \alpha_0}{\alpha_1 + \alpha_0}, \quad (3.6)$$

$$\sin \Theta = \sqrt{\frac{\alpha_1 - \alpha}{\alpha_1 - \alpha_0}}, \quad (3.7)$$

with  $\alpha = \cosh 2\chi$ ,  $\alpha_0 = \cosh 2\chi_0$ ,  $\alpha_1 = \cosh 2\chi_1$ . The shape above, supplemented by the closedness condition on the string i.e.  $\Delta\phi = \frac{2\pi}{n}$ ,  $n \in \mathbb{Z}$ , gives the total string profile. Here  $n$  is the total number of spikes formed. The angle between the valleys and the spikes is given by,

$$\Delta\phi = 2 \int_{\chi_0}^{\chi_1} \frac{\sinh 2\chi_0}{\sinh 2\chi} \frac{\sqrt{\cosh^2 \chi - \omega^2 \sinh^2 \chi}}{\sqrt{\sinh^2 2\chi - \sinh^2 2\chi_0}} d\chi \quad (3.8)$$

Again, the change of variables as before gives the solution

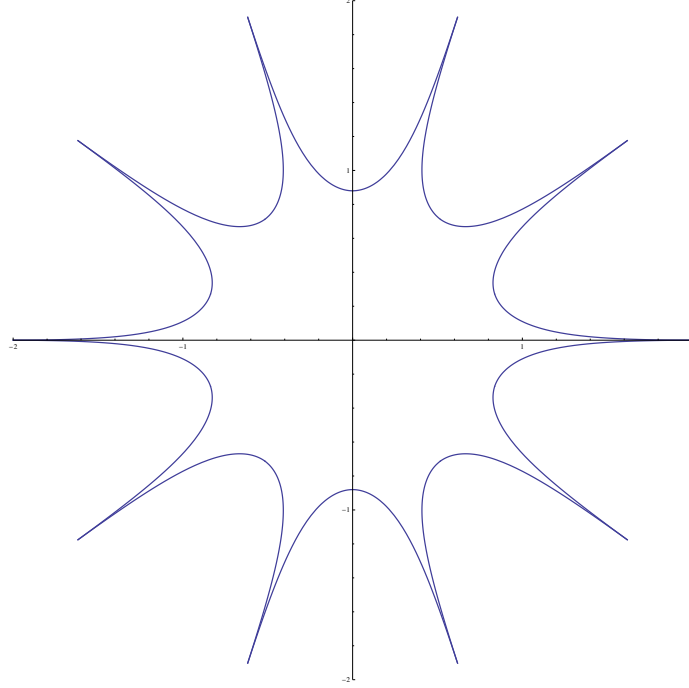
$$\Delta\phi = \frac{\sinh 2\chi_0}{\sqrt{2} \sinh \chi_1} \frac{1}{\sqrt{\alpha_0 + \alpha_1}} \left\{ \Pi \left( \frac{\alpha_1 - \alpha_0}{\alpha_1 - 1}, \frac{\pi}{2}, \mathcal{Q} \right) - \Pi \left( \frac{\alpha_1 - \alpha_0}{\alpha_1 + 1}, \frac{\pi}{2}, \mathcal{Q} \right) \right\}. \quad (3.9)$$

Clearly  $\Delta\phi$  is dependent on the parameters  $\chi_1$  and  $\chi_0$ . Now to implement the closed string condition we fix the height of the tip of spike by adjusting  $\omega$ . Following [56] we fix the spike height at  $\chi_1 = 2$  and  $n = 10$ . Then from the closeness condition, solving for  $\chi_0$  we get  $\chi_0 \simeq 0.88$ . Using these values (3.5) gives the plot of one spike, which we can rotate and reflect to plot the others. Note that the spikes will reach the AdS boundary if we make  $\chi_1 \rightarrow \infty$  i.e.  $\omega \rightarrow 1$ . Spiky strings in this ‘long’ string limit has been discussed in [56]. The total plot of the spikes is shown in the figure 9.

### 3.2 Nambu-Goto spiky strings in (AdS) $_{\kappa}$

We start with the one parameter deformed AdS<sub>3</sub> background, and write it using the coordinate transformations discussed earlier. The metric as used in [44] is

$$ds^2 = -\cosh^2 \chi dt^2 + \frac{1}{1 + \kappa^2 \cosh^2 \chi} (d\chi^2 + \sinh^2 \chi d\phi^2). \quad (3.10)$$



**Figure 9.** Spiky strings in AdS with 10 spikes, parameters  $\chi_1 = 2.0$  and  $\chi_0 = 0.88$ .

Here we use the same embedding ansatz as the simple AdS case i.e.,

$$t = \tau, \quad \phi = \omega\tau + \sigma, \quad \chi = \chi(\sigma). \quad (3.11)$$

Now as before, we will use the Nambu-Goto action to write the equations of motion of the fundamental string. Simply following the equations of motion for  $t$  and  $\phi$  we can write,

$$\sinh^2 \chi \cosh^2 \chi = C \sqrt{(\chi'^2 \cosh^2 \chi + \sinh^2 \chi \cosh^2 \chi)(1 + \varkappa^2 \cosh^2 \chi) - \omega^2 \chi'^2 \sinh^2 \chi}, \quad (3.12)$$

where  $C$  is a integration constant. Now this can be shown to satisfy the equation of motion for  $\chi$ , which we write as,

$$\chi' = \frac{1}{2} \frac{\sinh 2\chi}{\sinh 2\chi_0} \frac{\sqrt{\sinh^2 2\chi - (1 + \varkappa^2 \cosh^2 \chi) \sinh^2 2\chi_0}}{\sqrt{(1 + \varkappa^2 \cosh^2 \chi) \cosh^2 \chi - \omega^2 \sinh^2 \chi}}. \quad (3.13)$$

Here the integration constant is taken as before  $C^2 = \frac{1}{4} \sinh^2 2\chi_0$ . Now the above equation can be arranged to write the string profile as

$$\sigma = \varkappa \sinh 2\chi_0 \int \frac{1}{\sinh 2\chi} \frac{\sqrt{(\cosh 2\chi - \cosh 2\chi_1^+)(\cosh 2\chi - \cosh 2\chi_1^-)}}{\sqrt{(\cosh 2\chi - \cosh 2\chi_0^+)(\cosh 2\chi - \cosh 2\chi_0^-)}} d\chi, \quad (3.14)$$

where one can notice that the  $\chi'$  blows up for two values of  $\chi$ , they are

$$\chi_1^\pm = \frac{1}{2} \cosh^{-1} \frac{(\omega^2 - \varkappa^2 - 1) \pm \sqrt{1 - 2(1 + 2\varkappa^2)\omega^2 + \omega^4}}{\varkappa^2}, \quad (3.15)$$

so that we can say  $\chi_1$  represents the analogue to spike height of the corresponding string profile in this geometry. Note that only  $\chi_1^-$  reduces to the original spike height in AdS when we take the  $\varkappa \rightarrow 0$  limit in the proper way. The other root will not be interesting here.

Obviously, the “valleys” of the string profile now occurs at  $\chi_0^\pm$  which we get from the following expression:

$$\chi_0^\pm = \frac{1}{2} \cosh^{-1} \frac{\varkappa^2 \sinh^2 2\chi_0 \pm \sqrt{\varkappa^4 \sinh^4 2\chi_0 + (8\varkappa^2 + 16) \sinh^2 2\chi_0 + 16}}{4} \quad (3.16)$$

Now integrating the equation (3.14) we can write the total string profile as

$$\begin{aligned} \frac{2\sigma}{\varkappa \sinh 2\chi_0} &= \frac{2(\cosh 2p - \cosh 2r)}{\cosh^2 2r - 1} \sqrt{\frac{\cosh 2q - \cosh 2r}{\cosh 2p - \cosh 2s}} \mathbb{F}(\beta, \nu) \\ &+ \frac{(\cosh 2r - \cosh 2s)}{\sqrt{(\cosh 2p - \cosh 2s)(\cosh 2q - \cosh 2r)}} \left[ \frac{(\cosh 2p - 1)(\cosh 2q - 1)}{(\cosh 2r - 1)(\cosh 2s - 1)} \Pi(\gamma_-, \beta, \nu) \right. \\ &\quad \left. - \frac{(1 + \cosh 2p)(1 + \cosh 2q)}{(1 + \cosh 2r)(1 + \cosh 2s)} \Pi(\gamma_+, \beta, \nu) \right]. \end{aligned} \quad (3.17)$$

Here  $\mathbb{F}$  and  $\Pi$  are the usual incomplete Elliptic integrals of the second and third kind. We have also introduced the notation  $p = \chi_1^+$ ,  $q = \chi_1^-$ ,  $r = \chi_0^-$ ,  $s = \chi_0^+$  in the above equations for simplicity. Others symbols are defined as follows,

$$\beta = \sin^{-1} \sqrt{\frac{(\cosh 2q - \cosh 2r)(\cosh 2s - \cosh 2\chi)}{(\cosh 2q - \cosh 2s)(\cosh 2r - \cosh 2\chi)}}, \quad (3.18)$$

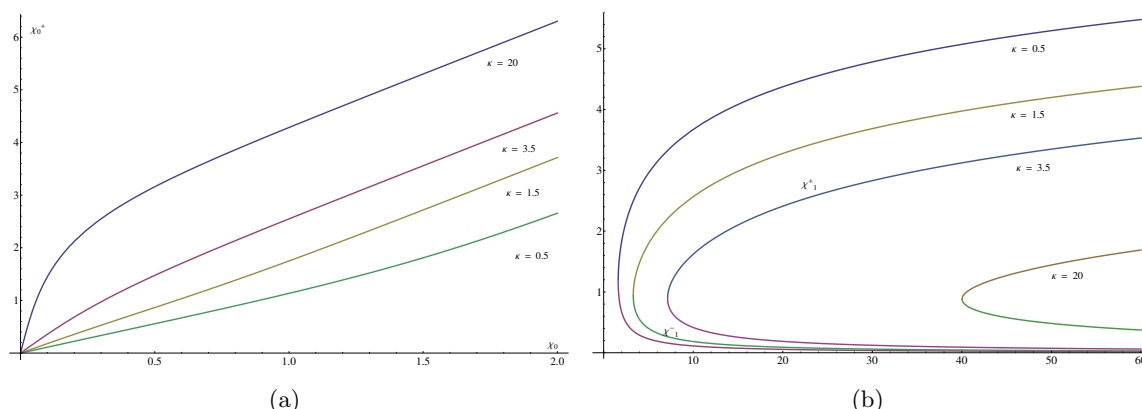
$$\nu = \frac{(\cosh 2p - \cosh 2r)(\cosh 2q - \cosh 2s)}{(\cosh 2q - \cosh 2r)(\cosh 2p - \cosh 2s)}, \quad (3.19)$$

$$\gamma_\pm = \frac{(\cosh 2r \pm 1)(\cosh 2q - \cosh 2s)}{(\cosh 2q - \cosh 2r)(\cosh 2s \pm 1)}. \quad (3.20)$$

Now we find the corresponding angle between valley and tip of the spike by choosing proper limits of the integral (3.14) since we can see that  $\cosh 2r$  is always negative and  $\cosh 2p > \cosh 2q > \cosh 2s$ . With suitable choice of limits we evaluate the integral as:

$$\begin{aligned} \frac{2\Delta\phi}{\varkappa \sinh 2\chi_0} &= \frac{2(\cosh 2p - \cosh 2r)}{\cosh^2 2r - 1} \sqrt{\frac{\cosh 2q - \cosh 2r}{\cosh 2p - \cosh 2s}} \mathbb{F}\left(\frac{\pi}{2}, \nu\right) \\ &+ \frac{(\cosh 2r - \cosh 2s)}{\sqrt{(\cosh 2p - \cosh 2s)(\cosh 2q - \cosh 2r)}} \left[ \frac{(\cosh 2p - 1)(\cosh 2q - 1)}{(\cosh 2r - 1)(\cosh 2s - 1)} \Pi\left(\gamma_-, \frac{\pi}{2}, \nu\right) \right. \\ &\quad \left. - \frac{(1 + \cosh 2p)(1 + \cosh 2q)}{(1 + \cosh 2r)(1 + \cosh 2s)} \Pi\left(\gamma_+, \frac{\pi}{2}, \nu\right) \right]. \end{aligned} \quad (3.21)$$

Here one can see that the expression has been integrated from  $\chi = \chi_0^+$  to  $\chi = \chi_1^-$  following our original notation. Since the values where the tip and ‘valley’ of the spikes are supposed to occur are complex functions of the parameters, the presence of desired solutions are also likely to be constrained by the parameter space of  $(\varkappa, \omega, \chi_0)$ . As the parameter space here is substantially bigger than the case in the previous section, we can get different classes of string profiles which can be smoothly deformed into one another by simply tweaking the parameters.



**Figure 10.** (a)  $\chi_0^+$  plotted for different values of  $\kappa$  against increasing  $\chi_0$ . (b)  $\chi_1^+$  plotted for different values of  $\kappa$  against increasing  $\omega$ .

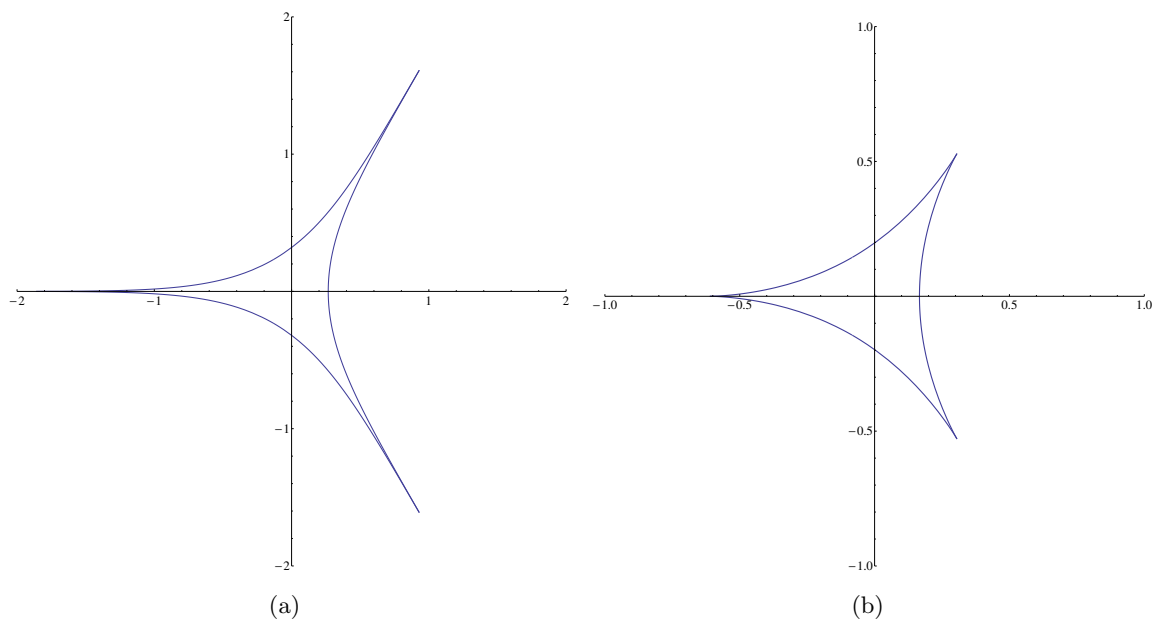
### 3.2.1 String profiles

Now we want to show the spiky profiles graphically and classify them. First, let us start examining the structure of the positive roots  $\chi = \chi_1^\pm, \chi_0^+$ . We plot the roots in the figure 10 for different values of  $(\kappa, \omega, \chi_0)$ . It is notable that (figure 10(b)) with increasing  $\omega$ , the plots for  $\chi_1^\pm$  shifts along the  $\omega$  axis. It is also important that the value of upper limit of integration  $\chi_1^-$  approaches zero asymptotically. To make the spike height very large it seems we have to take  $\omega \rightarrow 1$  and  $\kappa \rightarrow 0$  simultaneously, which is obviously the limit for the original AdS solution. One can also see that for small values of  $\kappa$  the lower limit of integration  $\chi_0^+$  varies almost linearly with the parameter  $\chi_0$ . As  $\chi_0 \rightarrow 0$  we can see  $\chi_0^+ \rightarrow 0$  also. These facts will be relevant when we will discuss the dynamics of the spikes in the long string limit. This more or less constrains the values of the parameter space where our desired solutions can form.

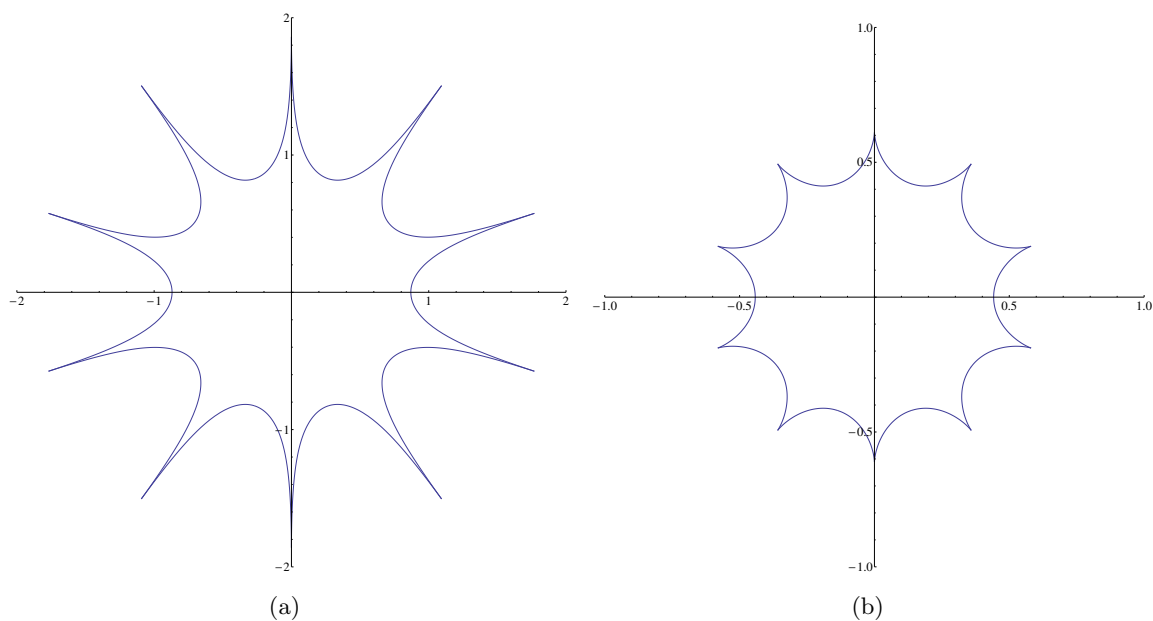
We can fix the values  $\chi_1^\pm$  by fixing the values of  $(\omega, \kappa)$  by hand. As described in the previous section, we can then extract the value of  $\chi_0$  imposing the closed string condition  $\Delta\phi = \frac{2\pi}{2n}$ ,  $n \in \mathbb{Z}$  on the string. This fixes all the parameters in the calculation and the polar plot seemingly gives us the spiky string solutions. Spiky strings for  $n = 3$  and  $n = 10$  are plotted in figures 11 and 12 for varying values of the parameters. Notice while the total profiles look more or less the same, the nature for the single segment has changed here. This can be understood by drawing tangents to the single segments and comparing with the undeformed case. One can actually verify that for the  $t = \tau = \text{constant}$  the induced metrics on the worldsheet section for  $\text{AdS}_3$  and  $(\text{AdS}_3)_\kappa$  can be related by a conformal factor of  $(1 + \kappa^2 \cosh^2 \chi)^{-1}$ . In that sense the spiky string solutions on the worldsheet can also be related to each other. It can also be noticed that as  $\kappa$  becomes smaller, the spikes reach out more to the apparent boundary. However, we found that for very high values of  $\kappa$  the closedness condition does not admit a viable solution for  $\chi_0$ .

A second kind of string solution is generated when we have  $\chi_0^+ > \chi_1^-$ , i.e. the so-called valleys form at higher points than the spikes. For these values in the usual undeformed problem, we should have got ‘dual’ spikes [62] where spikes form facing the AdS origin



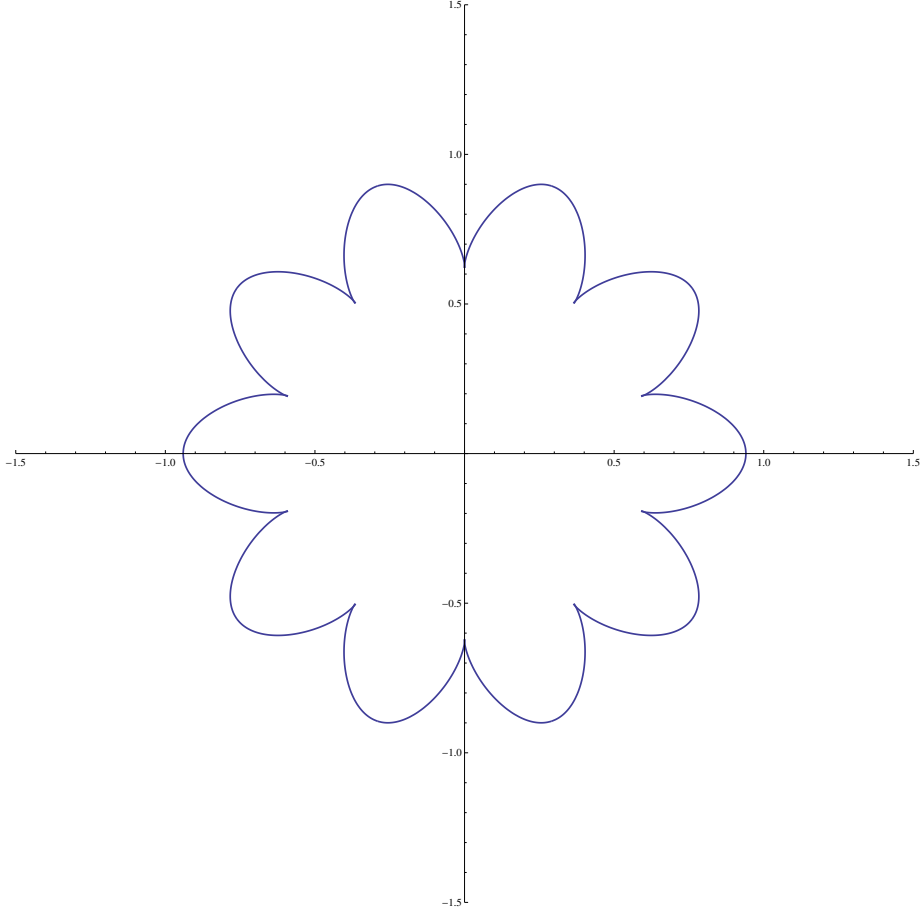


**Figure 11.** Closed string profiles with  $n = 3$  for various values of  $\kappa$ . The parameter values are (a)  $(\kappa, \omega) = (0.0001, 1.05)$ , (b)  $(\kappa, \omega) = (10.0, 22.0)$ .



**Figure 12.** Closed string profiles with  $n = 10$  for various values of  $\kappa$ . The parameter values are (a)  $(\kappa, \omega) = (0.0001, 1.05)$ , (b)  $(\kappa, \omega) = (10.0, 22.0)$ .

instead of approaching the boundary. It is worth noting that the Energy-Spin dispersion relation for such solutions is different than that of usual long spiky strings. For such parameter values in our deformed case we show the profiles remain unchanged. As an example one can see figure 13.



**Figure 13.** Spiky nature of closed strings forming towards the  $(\text{AdS})_{\kappa}$  origin. Here the parameter values are  $(\kappa, \omega) = (2.0, 4.7)$ .

### 3.2.2 A long string limit

For completeness, we look at the isometries of the background and construct the usual conserved quantities: energy  $E$  and spin  $S$ . The expression for the conserved charges here lead to the following relation:

$$\begin{aligned}
 \frac{E - \omega S}{n} = \frac{\sqrt{\lambda} \sqrt{1 + \kappa^2}}{2\pi} & \left[ \frac{\cosh 2r - \cosh 2p}{4(\cosh 2m + \cosh 2r)} \sqrt{\frac{\cosh 2q - \cosh 2r}{\cosh 2p - \cosh 2s}} \mathbb{F}\left(\frac{\pi}{2}, \nu\right) \right. \\
 & + \frac{\cosh 2r - \cosh 2s}{4\sqrt{(\cosh 2p - \cosh 2s)(\cosh 2q - \cosh 2r)}} \Pi\left(\zeta, \frac{\pi}{2}, \nu\right) \\
 & \left. + \frac{\frac{(\cosh 2p + \cosh 2m)(\cosh 2q + \cosh 2m)}{(\cosh 2r + \cosh 2m)(\cosh 2s + \cosh 2m)}}{\sqrt{(\cosh 2p - \cosh 2s)(\cosh 2q - \cosh 2r)}} \Pi\left(\Gamma, \frac{\pi}{2}, \nu\right) \right], \quad (3.22)
 \end{aligned}$$

where we have,

$$\begin{aligned} m &= \frac{1}{2} \cosh^{-1} \left( 1 + \frac{2}{\varkappa^2} \right), \\ \Gamma &= \frac{(\cosh 2m + \cosh 2r)(\cosh 2q - \cosh 2s)}{(\cosh 2q - \cosh 2r)(\cosh 2m + \cosh 2s)}, \\ \zeta &= \frac{\cosh 2q - \cosh 2s}{\cosh 2q - \cosh 2r} \end{aligned} \quad (3.23)$$

The other symbols were defined before and  $n$  is the number of spikes as usual. Now, the above one is a complicated expression in terms of various elliptic functions, which prove to be quite tough to expand in particular limits. Also, we can see from (3.15) that to get a ‘long’ string limit of such solutions ( $\chi_1^- \rightarrow \infty$ ) with a fixed  $\omega$ , one must apparently take  $\varkappa \rightarrow 0$ . We refrain from discussing this from this point of view and present an alternative analysis up to first order.

To take the  $\chi_1^- \rightarrow \infty$ , we start with a redefinition of the coordinates with  $\sinh^2 \chi_1^- = z$  and

$$\cosh 2\chi_1^- = 2z + 1, \quad \sin u = \frac{\sinh \chi}{\sinh \chi_1^-}. \quad (3.24)$$

Now with the redefinitions we can solve for  $\omega$  from (3.15) to get

$$\omega = \frac{\sqrt{1+z}\sqrt{1+(z+1)\varkappa^2}}{\sqrt{z}}, \quad (3.25)$$

where we ignore the -ve solution. Now since we have  $\chi_1^+ > \chi_1^-$ , we can take the following assumption

$$\frac{\sinh^2 \chi_1^-}{\sinh^2 \chi_1^+} = 1 - \delta \quad \delta \ll 1. \quad (3.26)$$

Now it can be shown that the  $\delta \ll 1$  is necessary to take the long string limit. Solving the above equation with the definitions of  $\chi_1^\pm$  we can find that

$$\sinh^2 \chi_1^- = z = \frac{\sqrt{1+\varkappa^2}}{\varkappa} + \mathcal{O}(\delta), \quad (3.27)$$

which in turn gives

$$\omega = \varkappa + \sqrt{1+\varkappa^2} + \mathcal{O}(\delta^2). \quad (3.28)$$

In what follows we will work in the leading order in  $\delta$ . Now, since the charges are integrated from  $\chi_0^+$  to  $\chi_1^-$ , we will need the condition  $\chi_1^- \gg \chi_0^+$  for a long string stretching upto the singularity surface which means,

$$\sin u|_{\chi_0^+} \approx 0. \quad (3.29)$$

Now this can be achieved by taking  $\chi_0 \rightarrow 0$ , which is also evident from the figure 10(a). Under these considerations we can write the Energy  $E$  as follows,

$$\begin{aligned} \frac{E}{n} &\simeq \frac{\sqrt{\lambda}\sqrt{1+\varkappa^2}}{\pi} \int_{\chi_0^+}^{\chi_1^-} \frac{\cosh^2 \chi}{\sqrt{(1+\varkappa^2 \cosh^2 \chi) \cosh^2 \chi - \omega^2 \sinh^2 \chi}} d\chi \\ &= \frac{\sqrt{\lambda}}{\pi} \sqrt{z} \int_0^{\frac{\pi}{2}} \frac{\sqrt{1+z \sin^2 u}}{\sqrt{1-(1-\delta) \sin^2 u}} du, \end{aligned} \quad (3.30)$$

where we have used the fact that

$$\sinh^2 \chi_1^+ \sinh^2 \chi_1^- = \frac{1 + \varkappa^2}{\varkappa^2}. \quad (3.31)$$

Similarly we can write the spin of the spike as

$$\frac{S}{n} = \frac{\sqrt{\lambda}}{\pi} \sqrt{z} \omega \int_0^{\frac{\pi}{2}} \frac{z \sin^2 u}{(\sqrt{1 - (1 - \delta) \sin^2 u}) (\sqrt{1 + z \sin^2 u}) (1 + \varkappa^2 (1 + z \sin^2 u))} du. \quad (3.32)$$

While finding out the charges we have used that for a spiky strings with  $n$  cusps,

$$\int d\sigma = 2n \int_{\chi_0^+}^{\chi_1^-} \frac{d\chi}{\chi'} = 2\pi. \quad (3.33)$$

Now due to the presence of the factor  $\sqrt{1 - (1 - \delta) \sin^2 u}$  in denominator, the charges will diverge in the upper limit. We then choose the integration contour carefully and write the above two integrals in the form

$$\int_0^{\frac{\pi}{2}} I = \int_0^{\frac{\pi}{2} - 2\delta^{\frac{1}{4}}} I + \int_{\frac{\pi}{2} - 2\delta^{\frac{1}{4}}}^{\frac{\pi}{2}} I. \quad (3.34)$$

For the first part of the integration we can expand the integrals in orders of  $\delta$  as follows

$$\begin{aligned} I_{1E} &= \frac{\sqrt{\lambda}}{\pi} \sqrt{z} \left[ \int_0^{\frac{\pi}{2}} \frac{\sqrt{1 + z \sin^2 u}}{\cos u} + \mathcal{O}(\delta) \right] du, \\ I_{1S} &= \frac{\sqrt{\lambda}}{\pi} \sqrt{z} \omega \left[ \int_0^{\frac{\pi}{2}} \frac{z \sin^2 u}{\cos u (\sqrt{1 + z \sin^2 u}) (1 + \varkappa^2 (1 + z \sin^2 u))} + \mathcal{O}(\delta) \right] du. \end{aligned} \quad (3.35)$$

The charges now look analogous to the ones evaluated in [44]. We can evaluate the above integrals and expand them out in the form

$$\begin{aligned} \int_0^{\frac{\pi}{2} - 2\delta^{\frac{1}{4}}} I_{1E} &= \frac{\sqrt{\lambda}}{\pi} \sqrt{z} \left[ -\sqrt{z} \sinh^{-1} \sqrt{z} + \frac{1}{4} \sqrt{1+z} \ln \left( \frac{(1+z)^2}{\delta} \right) + \frac{(2z-1)}{3\sqrt{1+z}} \sqrt{\delta} + \mathcal{O}(\delta) \right] \\ \int_0^{\frac{\pi}{2} - 2\delta^{\frac{1}{4}}} I_{1S} &= \frac{\sqrt{\lambda}}{\pi} \sqrt{z} \omega \left[ -\frac{\sqrt{z} \sqrt{1+\varkappa^2}}{1+(1+z)\varkappa^2} \tanh^{-1} \left( \frac{\sqrt{z}}{\sqrt{1+z} \sqrt{1+\varkappa^2}} \right) + \frac{1}{4} \frac{z \ln \left( \frac{(1+z)^2}{\delta} \right)}{\sqrt{1+z} (1+(1+z)\varkappa^2)} \right. \\ &\quad \left. - \frac{z(4z^2 \varkappa^2 - 5(1+\varkappa^2) - z(2+\varkappa^2))}{3(1+z)^{3/2} (1+(1+z)\varkappa^2)^2} \sqrt{\delta} + \mathcal{O}(\delta) \right] \end{aligned}$$

Now the second part of the integral can be evaluated using a midpoint rule to be written as

$$\begin{aligned} I_{2E} &= \frac{\sqrt{\lambda}}{\pi} \sqrt{z} \left[ 2\sqrt{1+z} - \frac{(2z-1)}{3\sqrt{1+z}} \sqrt{\delta} + \mathcal{O}(\delta) \right] \\ I_{2S} &= \frac{\sqrt{\lambda}}{\pi} \sqrt{z} \omega \left[ \frac{2z}{\sqrt{1+z} (1+(1+z)\varkappa^2)} + \frac{z(4z^2 \varkappa^2 - 5(1+\varkappa^2) - z(2+\varkappa^2))}{3(1+z)^{3/2} (1+(1+z)\varkappa^2)^2} \sqrt{\delta} + \mathcal{O}(\delta) \right] \end{aligned}$$

Now one can see that when we add everything up,  $E$  and  $S$  does not contain any  $\mathcal{O}(\sqrt{\delta})$  term. Then we can combine the above expressions and use the expressions for  $z$  and  $\omega$  to find

$$\begin{aligned} \frac{E - (\varkappa + \sqrt{1 + \varkappa^2})S}{n} &= \frac{\sqrt{\lambda}}{\pi} \frac{\sqrt{1 + \varkappa^2}}{\varkappa} \left[ -\sinh^{-1} \left( \sqrt{\frac{\sqrt{1 + \varkappa^2}}{\varkappa}} \right) \right. \\ &\quad \left. + (\varkappa + \sqrt{1 + \varkappa^2}) \tanh^{-1} \left( \frac{1}{\sqrt{1 + \varkappa^2} \sqrt{1 + \frac{\varkappa}{\sqrt{1 + \varkappa^2}}}} \right) + \mathcal{O}(\delta) \right]. \end{aligned} \quad (3.36)$$

Now as in [60] we define the changed variables

$$w_0 \equiv \sqrt{1 + \frac{\varkappa}{\sqrt{1 + \varkappa^2}}}, \quad k_0 \equiv \sqrt{1 - \frac{\varkappa}{\sqrt{1 + \varkappa^2}}}. \quad (3.37)$$

With a little manipulation, we can rewrite the key expressions as

$$\begin{aligned} -\sinh^{-1} \left( \sqrt{\frac{\sqrt{1 + \varkappa^2}}{\varkappa}} \right) &= \frac{1}{2} \ln \left[ \frac{w_0 - 1}{w_0 + 1} \right], \\ \tanh^{-1} \left( \frac{1}{\sqrt{1 + \varkappa^2} \sqrt{1 + \frac{\varkappa}{\sqrt{1 + \varkappa^2}}}} \right) &= \frac{1}{2} \ln \left[ \frac{1 + k_0}{1 - k_0} \right]. \end{aligned} \quad (3.38)$$

Then we are led to the dispersion relation of the exact form as in [44, 60], provided we take  $n = 2$ , i.e. the solution reduces to that of a GKP folded string solution. The final expression is of the form,

$$\frac{E - \frac{w_0}{k_0} S}{n} = \frac{\sqrt{\lambda}}{2\pi} \frac{\sqrt{1 + \varkappa^2}}{\varkappa} \left[ \frac{w_0}{k_0} \ln \left[ \frac{1 + k_0}{1 - k_0} \right] + \ln \left[ \frac{w_0 - 1}{w_0 + 1} \right] + \mathcal{O}(\delta) \right]. \quad (3.39)$$

From the above expression one can note that in the  $\varkappa \rightarrow 0$  limit the right hand side actually diverges. To take the other extreme limit  $\varkappa \rightarrow \infty$  we define the scaled quantities

$$\mathcal{E} = \frac{E}{\frac{\sqrt{\lambda}}{2\pi} \sqrt{1 + \varkappa^2}}, \quad \mathcal{S} = \frac{S}{\frac{\sqrt{\lambda}}{2\pi} \sqrt{1 + \varkappa^2}}. \quad (3.40)$$

Now in the  $\varkappa \rightarrow \infty$  limit we perform the rescalings  $\mathcal{E} \rightarrow \mathcal{E}$  and  $\mathcal{S} \rightarrow \mathcal{S}/\varkappa$  to keep the charges finite. The dispersion relation here looks like,

$$\frac{\mathcal{E} - 2\mathcal{S}}{n} = 2\sqrt{2} + \ln(3 - 2\sqrt{2}) + \mathcal{O}(\delta). \quad (3.41)$$

Remember, in this limit the string sigma model is related to that of  $dS_3 \times H^3$  by T-duality. By making the maximal deformation, we can actually interpolate between positive and negative curvature target spaces. But the ramification of these string solutions are yet to be explored fully.

## 4 Summary and conclusion

In this work we have presented ‘spiky’ string solutions in  $\varkappa$  deformed AdS in details. After a brief discussion on the original solutions in undeformed AdS we find the corresponding string profiles in the one parameter deformed model and visualise them graphically. We have used here the new type of coordinate system where only the spacetime inside the singularity surface is described. Both polyakov strings with the most general type of embedding ansatz and Nambu-Goto type simple strings were discussed and classified here. It was found that in the first case for the Polyakov strings the spikes seem to ‘open up’ or become parallel as  $\varkappa$  is increased, the spike solutions of [56] remain more or less similar as far as the cusp structures are concerned. We have also found out the ‘Dual’ spike solutions forming towards the  $(\text{AdS})_\varkappa$  origin. All these new solutions conform to the fact that classical string solutions in this background can not be stretched to the actual boundary. In such cases the usual AdS/CFT dictionary might not work completely, but inside the singularity surface we hope to make some sense out of the usual string solutions. Although the nature of these solutions have to be understood in a more rigorous way to justify this claim.

There are many problems still to be addressed here. One can look at the holographic entanglement entropy in this background to find out whether the usual area law holds or not. The nature of the singularity surface may be investigated in even more details using the full supergravity solution. The most fascinating question that remains unanswered here is the dual gauge theory side which might be able to shed light on the behaviour of these classical strings. The dual theory, if any, is expected to be highly nonlocal and requires careful investigation.

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